# MATH 103B – Discussion Worksheet 6

## May 25, 2023

**Topics:** Finite extensions and algebraic closures (Judson 21.1)

When $K$ is a field, let $\overline{K}$ denote its algebraic closure.

**Problem 1.** Let $F$ be a field. Prove $\overline{\overline{F}} = \overline{F}$. That is, if $\alpha$ is a root of some polynomial with coefficients from $\overline{F}$, then $\alpha$ is also a root of some polynomial over $F$. In other words, the algebraic closure of an algebraically closed field is itself.
*Hint*: Instead of considering any particular polynomial, use the following fact from the previous homework (Problem 21.5.14): Let $K$ be an algebraic extension of $E$, and $E$ an algebraic extension of $F$. Then $K$ is algebraic over $F$.

**Problem 2.** Prove that if $E$ is a proper field extension of $\overline{F}$, then $E$ is transcendental over $F$.
*Hint*: Use Problem 1.

**Problem 3.** Prove that if $F$ is algebraically closed, then a polynomial of degree $n$ has exactly $n$ roots (counting multiplicities).
*Hint*: Induction.

**Problem 4.** Prove that there is no intermediate extension for $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. That is, there does not exist a field $K$ such that $\mathbb{Q} \subsetneq K \subsetneq \mathbb{Q}(\sqrt[3]{2})$.
*Hint*: Use Theorem 21.17 from Judson.

**Problem 5.** Find a basis for the field extension $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2})$ over $\mathbb{Q}$, where $\zeta_3$ is a primitive third root of unity. What is $[K : \mathbb{Q}]$? Which polynomial has $K$ as its splitting field?

**Problem 6.** Find 2 generators $\alpha$ and $\beta$ so that $\mathbb{Q}(\alpha, \beta)$ is the splitting field of the polynomial $x^n - a$ over $\mathbb{Q}$ where $a$ is square free (i.e. if $n^2|a$, then $n^2$ is a unit).
*Hint*: Use Problem 5 for inspirations.